M201 TI

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Mathematics: A Second Level Course

Supplementary Handbook for M201

Techniques of Integration

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1.3

Contents

Introduction

This booklet summarizes the main techniques of integration. It is referred to in various units of the course. *You are not expected to memorize any of the information in the booklet*, but you are expected to be able to use this information with the booklet in front of you. You will be allowed to take the booklet into the examination with you, therefore *do not write anything in it.*

The booklet is written in the Leibniz notation, rather than the function notation used in the Foundation Course. It covers only the practical side of the evaluation of integrals. The theoretical side is treated in *Unit M100 13, Integration II*, and elsewhere.

Further information about integration (together with many other useful formulas) will be found in handbooks such as

Tables of Integrals and Other Mathematical Data by H. B. Dwight (Macmillan, 1961).
Table of Integrals by R. G. Hudson and J. Lipka (Wiley, 1917).

I Leibniz Notation

This column is devoted to examples which are placed opposite the piece of text they illustrate.

1 Derivatives

If x is a variable in the domain of a real function f, then

$$\frac{df(x)}{dx} \quad \text{or} \quad \frac{d}{dx} f(x)$$

means $f'(x)$, the derivative of the function f at x.

If we want to give the variable x a particular numerical value, say 13, then for $f'(13)$, we write

$$\left[\frac{df(x)}{dx}\right]_{x=13} \quad \left(\text{NOT } \frac{df(13)}{d13}\right).$$

1 Derivatives

$\dfrac{dy}{dx}$ means $f'(x)$, where $y = f(x)$.

$\dfrac{d(x^2)}{dx}$ means $f'(x)$, where $f: x \longmapsto x^2$;

$$\text{i.e. } \frac{d(x^2)}{dx} = 2x.$$

$$\left[\frac{d(x^2)}{dx}\right]_{x=3} = [2x]_{x=3} = 6.$$

2 Definite Integrals

$$\int_a^b f(x)\, dx$$

means

$$\int_a^b f.$$

2 Definite Integrals

$$\int_a^b x^2\, dx \text{ means } \int_a^b x \longmapsto x^2;$$

$$\text{i.e. } \int_a^b x^2\, dx = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3$$

3 Indefinite Integrals

If x is a variable in the domain of an integrable real function f, whose domain is an interval A, then

$$\int f(x)\, dx$$

means $F(x)$ where F is a primitive function of f, i.e.

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

for all a, b in A.

3 Indefinite Integrals

$\int x^2\, dx$ means $\frac{1}{3}x^3 + c$ for all $x \in R$, where c is a real number.

4 Further Notation

$$[F(x)]_a^b$$

means $F(b) - F(a)$. Also, we are not restricted to the use of x as a label for a variable; thus

$$\int_a^b f(x)\, dx = \int_a^b f(u)\, du.$$

4 Further Notation

$$[\tfrac{1}{3}x^3]_{10}^{13} = \tfrac{1}{3}(13)^3 - \tfrac{1}{3}(10)^3$$

$$\int_a^b x^2\, dx = \int_a^b u^2\, du$$

II Rules of Integration

In the following rules, f and g are continuous functions, both having codomain R, and their domains, A and B respectively, are intervals of R.

1 The Fundamental Theorem of Calculus

$$\int f(x)\,dx = \text{some function, } F\text{, with domain } A \text{ whose derivative at } x \text{ is } f(x).$$

We call the expression $f(x)$ the *integrand* and the function F a *primitive* of f.

2 The Addition Rule

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad (x \in A \cap B)$$

3 The Scalar Multiplication Rule

If α is any real number, then

$$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx \qquad (x \in A)$$

4 Integration by Parts

In this rule, the derived functions of f and g must also be continuous. The rule is

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \qquad (x \in A \cap B)$$

(See *Unit M100 13*, Section 2.2.)

5 Integration by Substitution

(i) Forwards Method

With an integral of the form

$$\int f(g(x))g'(x)\,dx$$

we substitute u for $g(x)$, then $g'(x) = \frac{du}{dx}$ and we write du for $\frac{du}{dx}\,dx$.

This last step above is easily remembered and can be justified.

In the examples for Section II all the functions have domain R.

1 The Fundamental Theorem of Calculus

$$\int x^2\,dx = \tfrac{1}{3}x^3, \text{ since } \frac{d}{dx}(\tfrac{1}{3}x^3) = x^2$$

2 The Addition Rule

$$\int (\cos x + x^2)\,dx = \int \cos x\,dx + \int x^2\,dx$$

3 The Scalar Multiplication Rule

$$\int 3x^2\,dx = 3\int x^2\,dx$$

4 Integration by Parts

$$\int \underset{\substack{\uparrow\\ f(x)}}{x}\ \underset{\substack{\uparrow\\ g'(x)}}{\cos x}\,dx = \underset{\substack{\uparrow\\ f(x)}}{x}\ \underset{\substack{\uparrow\\ g(x)}}{\sin x} - \int \underset{\substack{\uparrow\\ g(x)}}{\sin x}.\ \underset{\substack{\uparrow\\ f'(x)}}{1}\,dx$$

5 Integration by Substitution

(i) Forwards Method

Find

$$\int \underset{\substack{\uparrow\\ f(g(x))}}{\cos(x^2)}\underset{\substack{\uparrow\\ g'(x)}}{2x}dx$$

Substitute u for x^2.

Then

$$\int \cos(x^2)2x\,dx = \int \cos u\,du$$

$$= \sin u$$

$$= \sin(x^2)$$

So

$$\int f(g(x))g'(x)\,dx = \int f(u(x))\frac{du}{dx}\,dx$$

$$= \int f(u)\,du \quad (u \in A \cap g(B)).$$

(See *Unit M100 13*, Section 2.4.)

This rule is particularly useful when

$$f\colon x \longmapsto \frac{1}{x} \qquad (x \in R^+);$$

for then we have

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{du}{u}$$

$$= \ln u$$

$$= \ln(g(x)).$$

(ii) Backwards Method

$$\int f(x)\,dx = \int f(h(u))h'(u)\,du \qquad (x \in A,\ u \in C)$$

We substitute $h(u)$ for x and $h'(u)du$ for dx (i.e. $\frac{dx}{du}\,du$ for dx), where h is a *one–one function* with domain C and image set A. The reason for h being one–one is that for the substitution to be unambiguous h must be a function rather than a mapping. Eventually we will want to express the result of the integration in terms of the original variable which means that the inverse of h must also be a function. Hence h must be a one–one function (see *Unit M100 1, Functions*, Section 2.6).

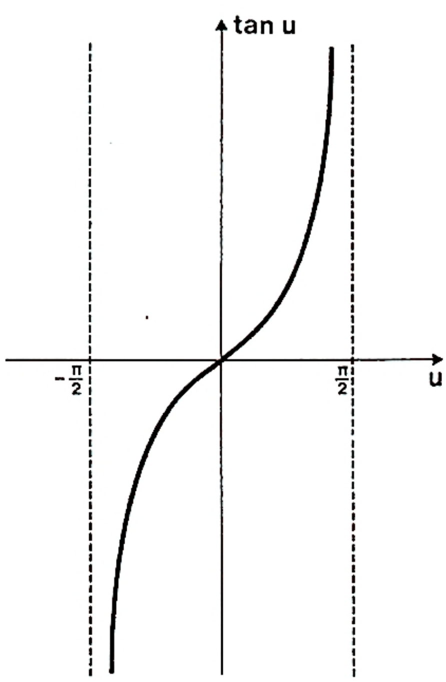

Fig 1

An important example of this rule of integration is

$$\int \frac{t}{t^2 + c}\,dt = \frac{1}{2}\int \frac{2t}{t^2 + c}\,dt$$

$$= \tfrac{1}{2}\ln(t^2 + c).$$

(ii) Backwards Method

$$\int \underbrace{\frac{1}{1 + x^2}}_{f(x)}\,dx = \int \underbrace{\frac{1}{1 + \tan^2 u}}_{f(\tan(u))}\ \underbrace{\sec^2 u}_{\tan' u}\,du$$

and since $1 + \tan^2 u = \sec^2 u$

$$\int \frac{1}{1 + x^2}\,dx = \int du$$

$$= u$$

$$= \arctan x.$$

In this case $h\colon u \longmapsto \tan u$, $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and arc tan is the inverse of h, i.e. arctan x is the radian measure of the angle in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ whose tangent is x.

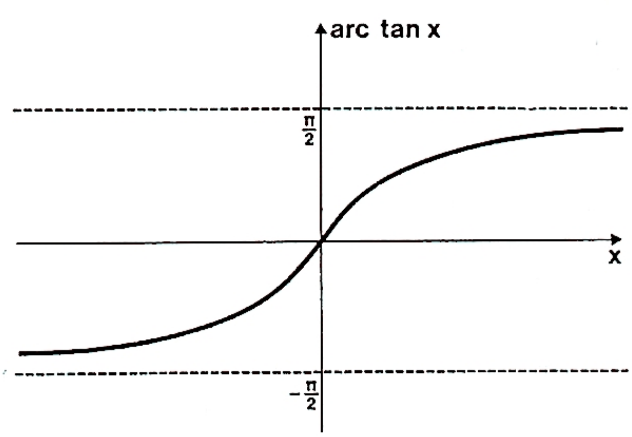

Fig 2

III Types of Integral

1 Polynomial Functions

To integrate a polynomial function, apply the addition rule and the scalar multiplication rule for integrals, to obtain

$$\int (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)\, dx$$

$$= a_n \frac{x^{n+1}}{n+1} + a_{n-1}\frac{x^n}{n} + \cdots + a_1 \frac{x^2}{2} + a_0 x$$

2 Rational Functions

A rational function has the form

$$x \longmapsto \frac{f(x)}{g(x)} \qquad (x \in R \text{ and } g(x) \neq 0)$$

where f and g are polynomial functions. We first give the integrals of some simple rational functions.

2.1 $$\int \frac{dx}{x-a} = \begin{cases} \ln(x-a) \text{ if domain } \subseteq \{x : x > a\} \\ \ln(a-x) \text{ if domain } \subseteq \{x : x < a\} \end{cases}$$

It is common practice to write the above formula as

$$\int \frac{dx}{x-a} = \ln|x-a|$$

but remember that this only makes sense if the domain includes numbers on one side of a only (since the integrand is not defined for $x = a$).

2.2

$$\int \frac{dx}{(x-a)^s} = \frac{-1}{(s-1)(x-a)^{s-1}} \quad \text{if} \quad s \neq 1$$

with the same restrictions on the domains as above.

This is perhaps more recognizable as

$$\int (x-a)^t\, dx = \frac{(x-a)^{t+1}}{t+1} \quad \text{where} \quad t = -s$$

2.3 For an integral of the form

$$\int \frac{px+q}{x^2+bx+c}\, dx,$$

where $b^2 < 4c$ (i.e. the denominator cannot be factorized), we first rewrite the denominator as the sum of two squares (known as "completing the square").

So $x^2 + bx + c$ becomes $(x + \frac{1}{2}b)^2 + c - \frac{1}{4}b^2$. We then use the substitution $t = x + \frac{1}{2}b$.

This removes the linear term from the denominator, i.e. it reduces the integral to the form

$$\int \frac{lt+m}{t^2+n}\, dt.$$

with $n > 0$.

1 Polynomial Functions

$$\int (3t^4 + 5t^3 - \sqrt{7})\, dt = 3\frac{t^5}{5} + 5\frac{t^4}{4} - \sqrt{7}\,t$$

2 Rational Functions

$$x \longmapsto \frac{x^3 - 3x + \sqrt{5}}{x+6} \qquad (x \in R,\ x \neq -6)$$

2.1 $$\int \frac{dx}{x} = \ln x \qquad (x \in R^+)$$

$$\int_{-2}^{-1} \frac{dx}{x} = [\ln(-x)]_{-2}^{-1}$$

$$= [\ln|x|]_{-2}^{-1}$$

$$= -\ln 2$$

whereas

$$\int_{-1}^{1} \frac{dx}{x} \text{ does } \textit{not} \text{ equal } [\ln|x|]_{-1}^{1}$$

In fact, this integral is not defined at all.

2.2

$$\int \frac{4\,dx}{(3x+2)^2} = \frac{4}{9}\int \frac{dx}{(x+\frac{2}{3})^2}$$

$$= -\frac{4}{9}\,\frac{1}{x+\frac{2}{3}} \begin{cases} (x > -\frac{2}{3}) \\ \text{or} \\ (x < -\frac{2}{3}) \end{cases}$$

2.3 To evaluate

$$\int \frac{x+1}{x^2+x+1}\, dx,$$

first complete the square in the denominator

$$\frac{x+1}{x^2+x+1} = \frac{x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}}$$

then use the substitution $t = x + \frac{1}{2}$.

Also, $\dfrac{dx}{dt} = 1$ so for dx we substitute dt.

Now use the two standard integrals

$$\int \frac{t\,dt}{t^2 + c} = \frac{1}{2}\ln(t^2 + c)$$

(See Section II.5(i) above.)

$$\int \frac{dt}{t^2 + c} = \frac{1}{\sqrt{c}}\arctan\left(\frac{t}{\sqrt{c}}\right)$$

(See Section II.5(ii) above.)

So
$$\int \frac{x+1}{x^2+x+1}\,dx = \int \frac{t+\frac{1}{2}}{t^2+\frac{3}{4}}\,dt$$
$$= \int \frac{t\,dt}{t^2+\frac{3}{4}} + \frac{1}{2}\int \frac{dt}{t^2+\frac{3}{4}}$$
$$= \tfrac{1}{2}\ln(t^2+\tfrac{3}{4})$$
$$+ \tfrac{1}{2}\sqrt{\tfrac{4}{3}}\arctan(t\sqrt{\tfrac{4}{3}})$$

2.4 We now give a general method for integrating rational functions.

(i) Express $\dfrac{f(x)}{g(x)}$ in the form

$$h(x) + \frac{k(x)}{g(x)}$$

where $h(x)$ and $k(x)$ are polynomials and $k(x)$ has degree (strictly) less than the degree of $g(x)$.

(See method opposite and *Units M201 9 and 29*.)

(ii) Factorize $g(x)$ into real factors which are either linear or quadratic (and irreducible).

(iii) Express $\dfrac{k(x)}{g(x)}$ as a sum of partial fractions.

(iv) Apply the addition rule for integrals and integrate each fraction according to the methods outlined above.

2.4 Suppose $\dfrac{2x^2+4}{2x-3} = \dfrac{f(x)}{g(x)}$.

The division goes as follows:

$$\begin{array}{rrl}
 & x + \frac{3}{2} & = h(x) \\
g(x) = 2x - 3 & \overline{)\,2x^2 + 0x + 4} & = f(x) \\
 & \underline{2x^2 - 3x} & = xg(x) \\
 & 3x + 4 & \\
 & \underline{3x - \frac{9}{2}} & = \frac{3}{2}g(x) \\
 & \frac{17}{2} & = k(x)
\end{array}$$

Thus $$\frac{2x^2+4}{2x-3} = x + \frac{3}{2} + \frac{17}{2(2x-3)}$$

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$$
$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

To determine the constants A, B and C, we reverse the process as follows

$$= \frac{Ax^2 + Ax + A + Bx^2 - Bx + Cx - C}{(x-1)(x^2+x+1)}$$
$$= \frac{(A+B)x^2 + (A-B+C)x + A - C}{x^3-1}$$

Since x^2, x and 1 are linearly independent polynomials, we can equate coefficients

$$A + B = 0$$
$$A - B + C = 0$$
$$A - C = 1$$

i.e.

$$A = \tfrac{1}{3},\ B = -\tfrac{1}{3},\ C = -\tfrac{2}{3}.$$

Thus

$$\frac{1}{x^3-1} = \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$$

$$\int \frac{1}{a^2-x^2}\,dx = \int \frac{1}{(a+x)(a-x)}\,dx$$

Step (ii)

$$= \int \frac{1}{2a}\left\{\frac{1}{(a+x)} + \frac{1}{(a-x)}\right\} dx$$

Step (iii)

$$= \frac{1}{2a}\{\ln(a+x) - \ln(a-x)\}$$

Step (iv)

$$= \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) \qquad (-a < x < a)$$

3 Algebraic Functions

3 Algebraic Functions

Integrals of the form

$$\int \frac{f(x) + h(x)\sqrt{ax^2+bx+c}}{g(x) + k(x)\sqrt{ax^2+bx+c}}\,dx$$

where $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are polynomials, occur quite frequently. They can be converted into integrals of rational functions using integration by substitution. We shall consider these substitutions in this section and for the sake of clarity we shall let the polynomials $f(x)$, $g(x)$, $h(x)$ and $k(x)$ take very simple forms.

3.1 If $a = 0$, the substitution

$$u = \sqrt{bx+c}$$

transforms the integrand to a rational function, to which the method of Section III.2 applies.

3.1 Find $\displaystyle\int \frac{dx}{1+\sqrt{x}}$.

The substitution is $u = \sqrt{x}$, i.e. $x = u^2$, giving

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+\sqrt{u^2}}\,2u\,du$$

$$= \int \frac{2u\,du}{1+u}$$

$$= 2u - 2\ln(1+u)$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x})$$

3.2 If $a \neq 0$, the substitution

$$u = x + \frac{b}{2a}$$

removes the "b" term inside the square root, giving an integral of the form

$$\int \frac{f_1(u) + h_1(u)\sqrt{(au^2+c_1)}}{g_1(u) + k_1(u)\sqrt{(au^2+c_1)}}\,du$$

3.2 Simplify $\displaystyle\int \frac{dx}{\sqrt{1+4x+5x^2}}$.

The substitution is $u = x + \frac{2}{5}$, i.e. $x = u - \frac{2}{5}$, and so

$$\int \frac{dx}{\sqrt{1+4x+5x^2}} = \int \frac{du}{\sqrt{5u^2+\frac{1}{5}}}$$

The next step depends on the signs of a and c_1.

(i) If $a > 0$ and $c_1 > 0$, the substitution

$$u = \frac{1}{2}\sqrt{\frac{c_1}{a}}\left(v - \frac{1}{v}\right) \qquad (v > 0)$$

transforms the integrand into a rational function.

Find $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$.

The substitution is

$$x = \frac{1}{2}\left(v - \frac{1}{v}\right)$$

with inverse $v = x + \sqrt{1+x^2}$ (since $v > 0$) and $\sqrt{1+x^2} = \frac{1}{2}\left(v + \frac{1}{v}\right)$. The rule of integration by substitution thus gives

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\frac{1}{2}\left(v+\frac{1}{v}\right)}\,\frac{1}{2}\left(1+\frac{1}{v^2}\right)dv\\ &= \int \frac{dv}{v}\\ &= \ln v\\ &= \ln\left(x + \sqrt{1+x^2}\right)\end{aligned}$$

(ii) If $a > 0$ and $c_1 < 0$

The substitution

$$u = \pm\frac{1}{2}\sqrt{\frac{c_1}{a}}\left(v + \frac{1}{v}\right) \qquad (v \geqslant 1)$$

(where the sign is chosen to make sure that u is in the stated domain) transforms the integrand into a rational function.

Find $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$ $\qquad (x \geqslant 1)$.

Here the substitution is $x = \frac{1}{2}\left(v + \frac{1}{v}\right)$, giving

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-1}}\,dx &= \int \frac{1}{\frac{1}{2}\left(v-\frac{1}{v}\right)}\,\frac{1}{2}\left(1-\frac{1}{v^2}\right)dv \; (v \geqslant 1)\\ &= \int \frac{dv}{v}\\ &= \ln v\\ &= \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)\end{aligned}$$

(iii) If $a < 0$ then we must have $c_1 > 0$.

The most useful substitution is

$$u = \sqrt{\frac{c_1}{-a}}\sin v \qquad \left(v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

which gives

$$\sqrt{au^2 + c_1} = \sqrt{c_1}\cos v$$

and converts the integrand to a rational function in $\sin v$ and $\cos v$. The integration of such functions is considered in the next section.

Find $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$.

The substitution $x = \sin v$, $v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ gives

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\cos v}\cos v\,dv\\ &= \int dv\\ &= v\\ &= \arcsin x\end{aligned}$$

These are fractions in which both the numerator and the denominator are polynomials in $\cos x$, whose coefficients are all polynomials in $\sin x$.

4 Rational Fractions in $\sin x$ and $\cos x$

For example $\dfrac{\cos x + (\sin^3 x + \sin x)\cos^2 x}{\sin x}$

4.1 We give first a general method for such integrals, but you are advised to use it with caution; for it can lead to heavy manipulations. The methods given in Section III.4.2 are often quicker. If the domain is a subset of $[-\pi, \pi]$, the general method is to substitute

$$x = 2 \arc\tan t \qquad (t \in R)$$

so that

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$\dfrac{dx}{dt}$ is found by considering which expression, when integrated, will give $2 \arctan t$.

i.e. $\dfrac{dx}{dt} = \dfrac{2}{(1+t^2)}$ (See Section II.5(ii).)

The inverse substitution is

$$t = \tan \tfrac{1}{2}x \qquad (x \in [-\pi, \pi])$$

If the domain is not a subset of $[-\pi, \pi]$, it can be split into segments of the form $[2n\pi - \pi, 2n\pi + \pi]$ with n an integer, and each segment treated separately by the substitution,

$$x = 2n\pi + 2 \arctan t.$$

It may happen that you obtain a zero denominator in which case another method will have to be used.

4.1

Find $\displaystyle\int \operatorname{cosec} x \, dx \qquad (0 < x < \tfrac{\pi}{2})$

$$\int \frac{1}{\sin x}\,dx = \int \frac{1+t^2}{2t}\,\frac{2}{1+t^2}\,dt$$
$$= \int \frac{dt}{t}$$
$$= \ln t$$
$$= \ln(\tan \tfrac{1}{2}x) + c \qquad (\text{since } 0 < t)$$

Find $\displaystyle\int \sec x \, dx \qquad (0 < x < \tfrac{\pi}{2})$

$$\int \frac{1}{\cos x}\,dx = \int \frac{1+t^2}{1-t^2}\,\frac{2}{1+t^2}\,dt$$
$$= \int \frac{2}{1-t^2}\,dt$$
$$= \int \frac{1}{1-t}\,dt + \int \frac{1}{1+t}\,dt$$
$$= -\ln(1-t) + \ln(1+t)$$
$$= \ln\left(\frac{1+t}{1-t}\right) \quad \text{(See Section III.2.4.)}$$

There are two ways of simplifying this expression:

(i) $$\ln \frac{1+t}{1-t} = \ln\left(\frac{(1+t)^2}{1-t^2}\right)$$
$$= \ln\left(\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right)$$
$$= \ln(\sec x + \tan x)$$

(ii) Using the formula

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

the equivalent result

$$\ln\left(\frac{1+t}{1-t}\right) = \ln\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

can be found.

4.2 Integrals of the form

$$\int \sin^m a\theta \cos^n a\theta \, d\theta \qquad (\theta \in R)$$

can be reduced, by the substitution $x = a\theta$, to the form

$$a^{-1} \int \sin^m x \cos^n x \, dx \qquad (x \in R).$$

4.2

(i) If n is an odd integer (not necessarily positive), the substitution

$$y = \sin x,$$

gives

$$\cos^2 x = 1 - y^2$$

and

$$\cos x \, dx = dy$$

thus converting the integrand to a rational function of y.

(ii) If m is an odd integer, the substitution

$$y = \cos x$$

gives

$$\sin^2 x = 1 - y^2$$

and

$$\sin x \, dx = -dy$$

thus converting the integrand to a rational function of y.

(iii) If m and n are both even integers, the substitution

$$y = \tan x$$

gives

$$\frac{1}{1+y^2} = \cos^2 x,$$

$$\frac{y^2}{1+y^2} = \sin^2 x$$

and

$$\frac{dy}{1+y^2} = dx$$

thus converting the integrand to a rational function of y.

4.3 Alternative Methods

One of the arts of integration is the ability to spot a quick method of producing the result.

$$\int \sin^2 x \cos^3 x \, dx$$
$$= \int y^2(1-y^2) \, dy$$
$$= \tfrac{1}{3} y^3 - \tfrac{1}{5} y^5$$
$$= \tfrac{1}{3} \sin^3 x - \tfrac{1}{5} \sin^5 x$$

$$\int \sin^2 x \cos^{-2} x \, dx$$
$$= \int \frac{y^2}{1+y^2} \frac{1+y^2}{1} \frac{dy}{1+y^2}$$
$$= \int \frac{y^2}{1+y^2} \, dy$$
$$= \int \left(1 - \frac{1}{1+y^2}\right) dy$$
$$= y - \arctan y$$
$$= \tan x - x$$

Unfortunately, as the powers of sin and cos increase, the work involved also increases somewhat disproportionately.

4.3 Alternative Methods

$$\int \sec x \, dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx$$
$$= \log(\sec x + \tan x)$$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$
$$= \tan x - x$$

$$\int \tan^5 x = \int (\sec^2 x - 1)^2 \tan x \, dx$$
$$= \int \left(\sec^4 x \tan x - 2 \sec^2 x \tan x + \frac{\sin x}{\cos x}\right) dx$$
$$= \tfrac{1}{4} \sec^4 x - \tan^2 x - \ln \cos x$$

5 Integrals Involving Trigonometric and Exponential Functions

5.1 Integrals involving products of the form

$$(\sin ax \text{ or } \cos ax) \times (\sin bx \text{ or } \cos bx)$$

can be simplified using the formulas

$$\sin ax \sin bx = \tfrac{1}{2}\cos(a-b)x - \tfrac{1}{2}\cos(a+b)x$$

$$\cos ax \cos bx = \tfrac{1}{2}\cos(a-b)x + \tfrac{1}{2}\cos(a+b)x$$

$$\sin ax \cos bx = \tfrac{1}{2}\sin(a-b)x + \tfrac{1}{2}\sin(a+b)x$$

5.2 If n is a positive integer and a, b are real, we have

(i)

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} \qquad (x \in R)$$

(ii)

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad (x \in R)$$

(iii) $$\int e^{ax}x^n\,dx = \left(\frac{x^n}{a} - \cdots + (-1)^r\frac{n!}{(n-r)!}\frac{x^{n-r}}{a^{r+1}} + \cdots + (-1)^n\frac{n!}{a^{n+1}}\right)e^{ax} \qquad (x \in R)$$

(iv) $$\int x^n\cos ax\,dx$$

$$= \left(\frac{x^n}{a} - \frac{n(n-1)}{a^3}x^{n-2} + \cdots\right)\sin ax + \left(\frac{nx^{n-1}}{a^2} - \frac{n(n-1)(n-2)}{a^4}x^{n-3} + \cdots\right)\cos ax$$

(v) $$\int x^n\sin ax\,dx$$

$$= -\left(\frac{x^n}{a} - \frac{n(n-1)}{a^3}x^{n-2} + \cdots\right)\cos ax + \left(\frac{nx^{n-1}}{a^2} - \frac{n(n-1)(n-2)}{a^4}x^{n-3} + \cdots\right)\sin ax$$

(vi) $$\int a^x dx = \frac{a^x}{\ln a}$$ (see opposite)

5 Integrals Involving Trigonometric and Exponential Functions

5.2 $$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

on integrating by parts

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

on integrating by parts again.

Re-arranging the above expression gives

$$(a^2+b^2)\int e^{ax}\cos bx\,dx = ae^{ax}\cos bx + be^{ax}\sin bx$$

so

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

This is a technique which can very often be used on expressions involving a single cos or sin function. To integrate $\int x^3e^x\,dx$ one uses the rule given which can be verified as follows:

$$\int x^3e^xdx = x^3e^x - \int 3x^2e^xdx,$$

on integrating by parts,

$$= x^3e^x - 3x^2e^x + 6\int xe^xdx$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6\int e^xdx$$

$$= (x^3 - 3x^2 + 6x - 6)e^x$$

$$\int a^xdx = \int (e^{\ln a})^xdx$$ (See *Unit M100 7, Sequences and Limits I*, Section 4.3, if this is not obvious.)

$$= \int e^{(\ln a)x}dx$$

$$= \frac{1}{\ln a}e^{(\ln a)x}$$

$$= \frac{a^x}{\ln a}$$

5.3 Integrals involving logarithms and inverse trigonometric functions can often be simplified by substituting

$u =$ (the logarithm or inverse trigonometric image).

5.3

$$\int \ln x\, dx = \int \ln(e^u)e^u du \quad \text{by the substitution } u = \ln x$$

$$= \int ue^u\, du$$

$$= ue^u - \int e^u\, du \quad \text{on integrating by parts}$$

$$= (u-1)e^u$$

$$= (\ln x - 1)x$$

The above example shows that one can get involved in heavy working if one only follows the book of rules.

For

$$\int \ln x\, dx = \int \ln x\, 1\, dx$$

$$= x \ln x - \int x \frac{1}{x}\, dx \quad \text{on integrating by parts}$$

$$= x \ln x - x$$

5.4 Integrals involving expressions of the form e^{-ax^2} often arise in statistics and physics. The substitution

$$t = -ax^2$$

sometimes reduces these to type 5.2(iii). If the integral is of the form

$$\int e^{-ax^2} x^n dx$$

then if n is odd, it can be reduced by the above substitution to type 5.2(iii), but if n is even this is not possible.

5.4

$$\int e^{-x^2}x^3\, dx = \tfrac{1}{2} \int e^t t\, dt$$

$$= \tfrac{1}{2}e^t(t-1) = \tfrac{1}{2}e^{-x^2}(-x^2-1)$$

$$\int e^{-x^2}x^2\, dx = \pm \tfrac{1}{2} \int e^t \sqrt{t}\, dt$$

which cannot be expressed in terms of elementary functions such as exp, ln, arc tan, etc.

5.5 If the integral involves the functions tan, cot, sec, or cosec, reduce it to one of the types considered previously by using

$$\tan x = \frac{\sin x}{\cos x}, \ \cot x = \frac{\cos x}{\sin x},$$

$$\sec x = \frac{1}{\cos x}, \ \operatorname{cosec} x = \frac{1}{\sin x}$$

$(x \in R$, denominator $\neq 0)$

5.5 Simplify $\dfrac{(\operatorname{cosec}^2 x + \sec^2 x)\sin^5 x}{\cot^3 x}$

As $\sec^2 x = \operatorname{cosec}^2 x \tan^2 x$ and $\cot^3 x = \dfrac{\cos^3 x}{\sin^3 x}$,

$$\frac{(\operatorname{cosec}^2 x + \sec^2 x)\sin^5 x}{\cot^3 x}$$

$$= \operatorname{cosec}^2 x(1 + \tan^2 x)\frac{\sin^8 x}{\cos^3 x}$$

$$= \operatorname{cosec}^2 x \sec^2 x \sin^8 x \cos^{-3} x$$

(since $1 + \tan^2 x = \sec^2 x$)

$$= \sin^6 x \cos^{-5} x$$

The integral of this can now be found by the methods of Section III.4.1.

LINEAR MATHEMATICS